Knowing Mathematics for Teaching

Who Knows Mathematics Well Enough To Teach Third Grade, and How Can We Decide?

By Deborah Loewenberg Ball, Heather C. Hill, and Hyman Bass

With the release of every new international mathematics assessment, concern about U.S. students' mathematics achievement has grown. Each mediocre showing by American students makes it plain that the teaching and learning of mathematics needs improvement. Thus, the country, once more, has begun to turn its worried attention to mathematics education. Unfortunately, past reform movements have consisted more of effort than effect. We are not likely to succeed this time, either, without accounting for the disappointing outcomes of past efforts and examining the factors that contribute to success in other countries. Consider what research and experience consistently reveal: Although the typical methods of improving U.S. instructional quality have been to develop curriculum, and—especially in the last decade—to articulate standards for what students should learn, little improvement is possible without direct attention to the practice of teaching. Strong standards and quality curriculum are important. But no curriculum teaches itself, and standards do not operate independently of professionals' use of them. To implement standards and curriculum effectively, school systems depend upon the work of skilled teachers who understand the subject matter. How well teachers know mathematics is central to their capacity to use instructional materials wisely, to assess students' progress, and to make sound judgments about presentation, emphasis, and sequencing.

That the quality of mathematics teaching depends on teachers' knowledge of the content should not be a surprise. Equally unsurprising is that many U.S. teachers lack sound mathematical understanding and skill. This is to be expected because most teachers—like most other adults in this country—are graduates of the very system that we seek to improve. Their own opportunities to learn mathematics have been uneven, and often inadequate, just like those of their non-teaching peers. Studies over the past 15 years consistently reveal that the mathematical knowledge of many teachers is dismayingly thin.[1] Invisible in this research, however, is the fact that the mathematical knowledge of most adult Americans is as weak, and often weaker. We are simply failing to reach reasonable standards of mathematical proficiency with most of our students, and those students become the next generation of adults, some of them teachers. This is a big problem, and a challenge to our desire to improve.

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[1] For example, Liping Ma's 1999 book, *Knowing and Teaching Elementary Mathematics,* broadened interest in the question of how teachers need to know mathematics to teach (Ma, 1999). In her study, Ma compared Chinese and U.S. elementary teachers' mathematical knowledge. Producing a portrait of dramatic differences between the two groups, Ma used her data to develop a notion of "profound understanding of fundamental mathematics," an argument for a kind of connected, curricularly-structured, and longitudinally coherent knowledge of core mathematical ideas. (For a review of this book, see the Fall 1999 issue of *American Educator,* **www.aft.org/pubs-reports/american_educator/fall99/amed1.pdf.**)

What is less obvious is the remedy. One often-proposed solution is to require teachers to study more mathematics, either by requiring additional coursework for teachers,[2] or even stipulating a subject-matter major.[3] Others advocate a more practice-grounded approach, preparing teachers in the mathematics they will use on the job. Often, these advocates call for revamping mathematics methods coursework and professional development to focus more closely on the mathematics contained in classrooms, curriculum materials, and students' minds. Still others argue that we should draw new recruits from highly selective colleges, betting that overall intelligence and basic mathematics competence will prove effective in producing student learning. Advocates for this proposal pointedly eschew formal education courses for these new recruits, betting that little is learned in schools of education about teaching mathematics effectively.

At issue in these proposals is the scope and nature of the mathematical knowledge needed for teaching. Do teachers need knowledge of advanced calculus, linear algebra, abstract algebra, differential equations, or complex variables in order to successfully teach high school students? Middle school students? Elementary students? Or do teachers only need to know the topics they actually teach to students? Alternatively, is there a professional knowledge of mathematics *for teaching,* tailored to the work teachers do with curriculum materials, instruction, and students?

Despite the uproar and the wide array of proposed solutions, the effects of these advocated changes in teachers' mathematical knowledge on student achievement are unproven or, in many cases, hotly contested. Although many studies demonstrate that teachers' mathematical knowledge helps support increased student achievement, the actual nature and extent of that knowledge—whether it is simply basic skills at the grades they teach, or complex and professionally specific mathematical knowledge—is largely unknown. The benefits to student learning of teachers' additional coursework, either in mathematics itself or "mathematics methods"—courses that advise ways to teach mathematics to students—are disputed by leading authorities in the field. Few studies have been successful in pinpointing an appropriate mathematics "curriculum"—whether it be purely mathematical, grounded in practice, or both—that can provide teachers with the appropriate mathematics to help students learn (Wilson and Berne, 1999). Similarly, we know too little about the effectiveness of recruits who bypass traditional schools of education. What is needed are more programs of research that complete the cycle, linking teachers' mathematical preparation and knowledge to their students' achievement.

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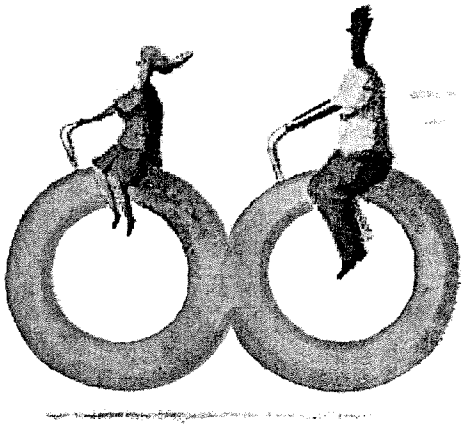

In this article, we describe one such program of research that we have been developing for more than a decade. In 1997, building on earlier work (see Ball and Bass, 2003), we

[2] For example, in the 2001 report, *The Mathematical Education of Teachers,* the Conference Board of the Mathematical Sciences, American Mathematical Society, and Mathematical Association of America call for prospective elementary teachers to take at least nine semester-hours on fundamental ideas of elementary school mathematics; prospective middle-grades math teachers to take at least 21 semester-hours of mathematics, including at least 12 semester-hours on fundamental ideas of school mathematics appropriate for the middle grades; and prospective high school mathematics teachers to complete the equivalent of an undergraduate major in mathematics, including a 6-hour capstone course connecting their college mathematics courses with high school mathematics. The report recommends that prospective teachers take mathematics courses "that develop a deep understanding of the mathematics they will teach," and "a thorough mastery of the mathematics in several grades beyond that which they expect to teach, as well as of the mathematics in earlier grades."

[3] NCLB requires that all new middle- and high-school teachers demonstrate subject-matter competency by 1) passing a state academic subject test in each of the subjects in which they teach; or 2) completing an academic major, a graduate degree, coursework equivalent to an undergraduate academic major, or advanced certification or credentialing in each of the subjects in which they teach (Public Law 107-110, Section 9101 [23]).

began a close examination of the actual work of teaching elementary school mathematics, noting all of the challenges in this work that draw on mathematical resources, and then we analyzed the nature of such mathematical knowledge and skills and how they are held and used in the work of teaching. From this we derived a practice-based portrait of what we call "mathematical knowledge for teaching"—a kind of professional knowledge of mathematics different from that demanded by other mathematically intensive occupations, such as engineering, physics, accounting, or carpentry. We then rigorously tested our hypothesis about this "professional" knowledge of mathematics, first by generating special measures of teachers' professional mathematical knowledge and then by linking those measures to growth in students' mathematical achievement. We found that teachers who scored higher on our measures of mathematical knowledge for teaching produced better gains in student achievement. This article traces the development of these ideas and describes this professional knowledge of mathematics for teaching.

What Does It Mean To Know Mathematics for Teaching?

Every day in mathematics classrooms across this country, students get answers mystifyingly wrong, obtain right answers using unconventional approaches, and ask questions: Why does it work to "add a zero" to multiply a number by ten? Why, then, do we "move the decimal point" when we multiply decimals by ten? And is this a different procedure or different aspects of the same procedure—changing the place value by one unit of ten? Is zero even or odd? What is the smallest fraction? Mathematical procedures that are automatic for adults are far from obvious to students; distinguishing between everyday and technical uses of terms—mean, similar, even, rational, line, volume—complicates communication. Although polished mathematical knowledge is an elegant and well-structured domain, the mathematical knowledge held and expressed by students is often incomplete and difficult to understand. Others can avoid dealing with this emergent mathematics, but teachers are in the unique position of having to professionally scrutinize, interpret, correct, and extend this knowledge.

Having taught and observed many mathematics lessons ourselves, it seemed clear to us that these "classroom problems" were also mathematical problems—but not the kind of mathematical problems found in the traditional disciplinary canons or coursework. While it seemed obvious that teachers had to know the topics and procedures they teach—factoring, primes, equivalent fractions, functions, translations and rotations, and so on—our experiences and observations kept highlighting additional dimensions of the knowledge useful in classrooms. In keeping with this observation, we decided to focus our efforts on bringing the nature of this additional knowledge to light, asking what, in practice, teachers need to know about mathematics to be successful with students in classrooms.

To make headway on these questions, we have focused on the "work of teaching" (Ball, 1993; Lampert, 2001). What do teachers *do* in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill? Instead of starting with the curriculum they teach, or the standards for which they are responsible, we have been studying teachers' work. By "teaching," we mean everything that teachers do to support the instruction of their students. Clearly we mean the interactive work of teaching lessons in classrooms, and all the tasks that arise in the course of that. But we also mean planning those lessons, evaluating students' work, writing and grading assessments, explaining class work to parents, making and managing homework, attending to concerns for equity, dealing with the building principal who has strong views about the math curriculum, etc. Each of these tasks involves knowledge of mathematical ideas, skills of mathematical reasoning and communication, fluency with examples and terms, and thoughtfulness about the nature of mathematical proficiency (Kilpatrick, Swafford, and Findell, 2001).

To illustrate briefly what it means to know mathematics for teaching, we take a specific mathematical topic—multiplication of whole numbers. One aspect of this knowledge is to be able to use a reliable algorithm to calculate an answer. Consider the following multiplication problem:

$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline \end{array}$$

Most readers will remember how to carry out the steps of the procedure, or algorithm, they learned, resulting in the following:

$$\begin{array}{r} {\scriptstyle 1} \\ {\scriptstyle 2} \\ 35 \\ \times\ 25 \\ \hline 175 \\ 70 \\ \hline 875 \end{array}$$

Clearly, being able to multiply correctly is essential knowledge for teaching multiplication to students. But this is also insufficient for teaching. Teachers do not merely do problems while students watch. They must explain, listen, and examine students' work. They must choose useful models or examples. Doing these things requires additional mathematical insight and understanding.

Teachers must, for example, be able to see and size up a typical wrong answer:

$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 175 \\ 70 \\ \hline 245 \end{array}$$

Recognizing that this student's answer as wrong is one step, to be sure. But effective teaching also entails analyzing the source of the error. In this case, a student has not "moved over" the 70 on the second line.

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Sometimes the errors require more mathematical analysis:

$$\begin{array}{r} \scriptstyle 1 \\ \scriptstyle 2 \\ 35 \\ \times\ 25 \\ \hline 255 \\ 80 \\ \hline 1055 \end{array}$$

What has happened here? Teachers may have to look longer at the mathematical steps that produced this, but most will be able to see the source of the error.[4] Of course teachers can always ask students to explain what they did, but if a teacher has 30 students and is at home grading students' homework, it helps to have a good hypothesis about what might be causing the error.

But error analysis is not all that teachers do. Students not only make mistakes, they ask questions, use models, and think up their own non-standard methods to solve problems. Teaching also involves explaining *why* the 70 should be slid over so that the 0 is under the 7 in 175—that the second step actually represents 35 × *20*, not 35 × *2* as it appears.

Teaching entails using representations. What is an effective way to represent the *meaning* of the algorithm for multiplying whole numbers? One possible way to do it is to use an area model, portraying a rectangle with side lengths of 35 and 25, and show that the area produced is 875 square units:

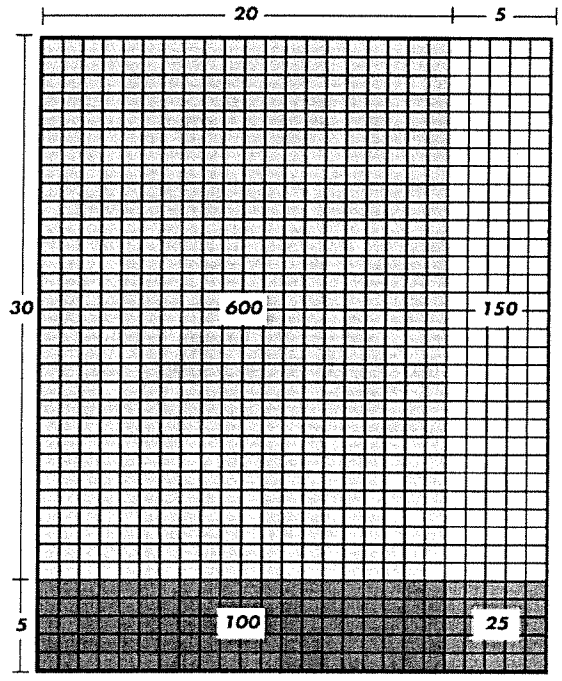


Figure 1.

Doing this carefully requires explicit attention to units, and to the difference between linear (i.e., side lengths) and area measures (Ball, Lubienski, and Mewborn, 2001).

Connecting Figure 1 to the full partial product version of the algorithm is another aspect of knowing mathematics for teaching:

$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 25 \\ 150 \\ 100 \\ +\ 600 \\ \hline 875 \end{array}$$

The model displays each of the partial products—25, 150, 100, and 600—and shows the factors that produce those products—5 × 5 (lower right hand corner), 20 × 5 (lower left hand corner), for example. Examining the diagram vertically reveals the two products—700 and 175—from the conventional algorithm illustrated earlier:

$$\begin{array}{r} 35 \\ \times\ 25 \\ \hline 175 \\ 70 \\ \hline 875 \end{array}$$

Representation involves substantial skill in making these connections. It also entails subtle mathematical considerations. For example, what would be strategic numbers to use in an example? The numbers 35 and 25 may not be ideal choices to show the essential conceptual underpinnings of the algorithm. Would 42 and 70 be better? What are the considerations in choosing a good example for instructional purposes? Should the numerical examples require regrouping, or should examples be sequenced from ones requiring no regrouping to ones that do? And what about the role of zeros at different points in the procedure? Careful advance thought about such choices is a further form of mathematical insight crucial to teaching.[5]

Note that nothing we have said up to this point involves knowing about students. Nothing implies a particular way to teach multiplication or to remedy student errors. We do not suggest that such knowledge is unimportant. But we do argue that, in teaching, there is more to "knowing the subject" than meets the eye. We seek to uncover what that "more" is. Each step in the multiplication example has involved a deeper and more explicit knowledge of multiplica-

[4] Here the student has likely multiplied 5 × 5 to get 25, but then when the student "carried" the 2, he or she added the 2 to the 3 before multiplying it by the 5—hence, 5 × 5 again, yielding 25, rather than (3 × 5) + 2 = 17. Similarly, on the second row, he or she added the 1 to the 3 before multiplying, yielding 4 × 2 instead of (3 × 2) + 1 = 7.

[5] Two-digit factors, with "carries," present all general phenomena in the multiplication algorithm in computationally simple cases. The presence of zero digits in either factor demands special care. The general rules still apply, but because subtleties arise, these problems are not recommended for students' first work. For example, in 42 × 70, students must consider how to handle the 0. In general, it is preferable for students to master the basic algorithm (i.e., multiplication problems with no regrouping) before moving on to problems that present additional complexities.

tion than that entailed by simply performing a correct calculation. Each step points to some element of knowing the topic in ways central to teaching it.

Our example helps to make plain that knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably. Further, it indicates that there are predictable and recurrent tasks that teachers face that are deeply entwined with mathematics and mathematical reasoning—figuring out where a student has gone wrong (error analysis), explaining the basis for an algorithm in words that children can understand and showing why it works (principled knowledge of algorithms and mathematical reasoning), and using mathematical representations. Important to note is that each of these common tasks of teaching involves *mathematical* reasoning as much as it does pedagogical thinking.

We deliberately chose an example involving concepts of number and operations. Similar examples can be developed about most mathematical topics, including the definition of a polygon (Ball and Bass, 2003), calculating and explaining an average, or proving the completeness of a solution set to an elementary mathematics problem. Being able to carry out and understand multi-step problems is another site for explicit mathematical insight in teaching. Each of these requires more than being able to answer the question oneself. The teacher has to think from the learner's perspective and to consider what it takes to understand a mathematical idea for someone seeing it for the first time. Dewey (1902) captured this idea with the notion of "psychologizing" the subject matter, seeing the structures of the subject matter as it is learned, not only in its finished logical form.

It should come as no surprise then that an emergent theme in our research is the centrality of mathematical language and the need for a special kind of fluency with mathematical terms. In both our records from a variety of classrooms and our experiments in teaching elementary students, we see that teachers must constantly make judgments about how to define terms and whether to permit informal language or introduce and use technical vocabulary, grammar, and syntax. When might imprecise or ambiguous language be pedagogically preferable and when might it threaten the development of correct understanding? For example, is it fair to say to second-graders that they "cannot take a larger number away from a smaller one" or does concern for mathematical integrity demand an accurate statement (for example, "with the numbers we know now, we do not have an answer when we subtract a large number from a smaller one")? How should a rectangle be defined so that fourth-graders can sort out which of the shapes in Figure 2 are and are not called "rectangles," and why?

The typical concept held by fourth-graders would lead them to be unsure about several of these shapes, and the commonly-held "definition"—"a shape with two long sides and two short sides, and right angles"—does not help them to reconcile their uncertainty. Students who learn shapes only by illustration and example often construct images that are entirely wrong. For example, in a fourth-grade class taught by Ball, several students believed that "A" in Figure 2 was a rectangle because it was a "box," and, in an age of

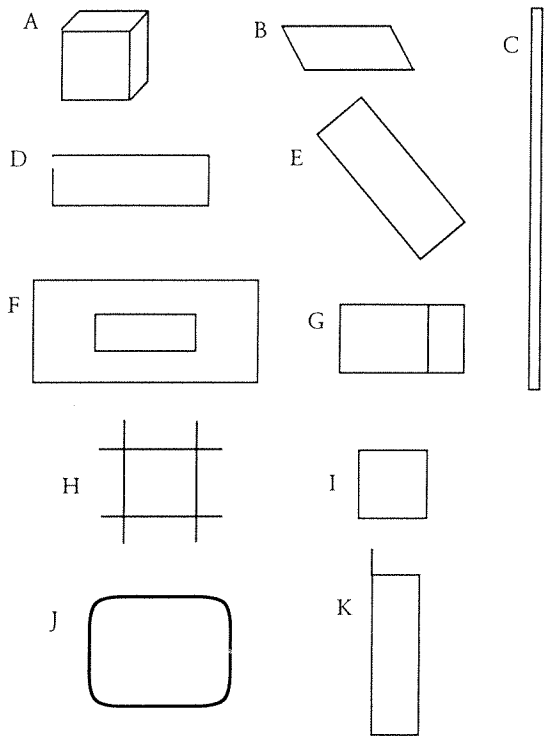
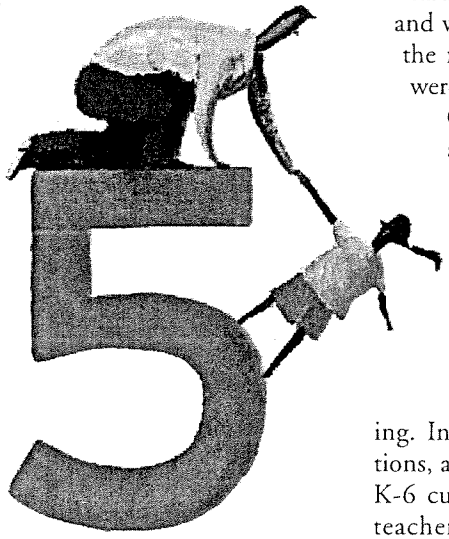


Figure 2. Candidate shapes: Which are rectangles?

computer graphics, they translated "rectangle" to "box" without a blink. Teachers need skill with mathematical terms and discourse that enable careful mathematical work by students, and that do not spawn misconceptions or errors. Students need definitions that are usable, relying on terms and ideas they already understand. This requires teachers to know more than the definitions they might encounter in university courses. Consider, for example, how "even numbers" might be specified for learners in ways that do not lead students to accept 1½ as even (i.e., it can be split into two equal parts) and, still, to identify zero as even. For example, defining even numbers as "numbers that can be divided in half equally" allows ¼, 1½, ⅕, and all other fractions to be considered even. Being more careful would lead to definitions such as, "A number is considered even if and only if it is the sum of an integer with itself" or, for students who do not work with integers yet: "Whole numbers that can be divided into pairs (or twos) with nothing left over are called *even* numbers." Although expressed in simpler terms, these definitions are similar to a typical definition taught in number theory: "Even numbers are of the form 2k, where k is an integer." They are accessible to elementary students without sacrificing mathematical precision or integrity.

In our data, we see repeatedly the need for teachers to have a specialized fluency with mathematical language, with what counts as a mathematical explanation, and with how to use symbols with care. In addition to continuing to probe the ways in which teachers need to understand the topics of the school curriculum, and the mathematical ideas to which they lead, we will explore in more detail how mathematical language—its construction, use, and cultivation—is used in the work of teaching.

Knowing mathematics for teaching demands a kind of depth and detail that goes well beyond what is needed to carry out the algorithm reliably.



Measuring Mathematical Knowledge for Teaching

Using the methods described above, we could have continued simply to explore and map the terrain of mathematical knowledge for teaching. Because such work is slow and requires great care, we examined only a fraction of the possible topics, grade levels, and mathematical practices teachers might know. However, we believe that *only* developing grounded theory about the elements and definition of mathematical knowledge for teaching is not enough. If we argue for professional knowledge for teaching mathematics, the burden is on us to demonstrate that improving this knowledge also enhances student achievement. And, as the current debates over teacher preparation demonstrate, there are legitimate competing definitions of mathematical knowledge for teaching and, by extension, what "teacher quality" means for mathematics instruction. To test our emerging ideas, and provide evidence beyond examples and logical argument, we developed (and continue to refine) large-scale survey-based measures of mathematical knowledge for teaching.

Our two main questions were: Is there a body of mathematical knowledge for teaching that is specialized for the work that teachers do? And does it have a demonstrable effect on student achievement? To answer these questions, we needed to build data sets that would allow us to test our hypotheses empirically. This required us to pose many items to a large number of teachers; to control for the many factors that are also likely to contribute to students' learning and detect an effect of what we hypothesized as "mathematical knowledge for teaching," large data sets were essential. Anticipating that samples of a thousand or more teachers might be required to answer our questions, however, we quickly saw that interviews, written responses, and other forms of measuring teachers' mathematical knowledge would not do, and we set out to try to develop multiple-choice measures, the feasibility of which others doubted and we ourselves were unsure.

Our collaborators experienced in educational measurement informed us that the first step in constructing any assessment is to set out a "domain map," or a description of the topics and knowledge to be measured. We chose to focus our initial work within the mathematical domains that are especially important for elementary teaching: number and operations. These are important both because they dominate the school curriculum and because they are vital to students' learning. In addition, we chose the domain of patterns, functions, and algebra because it represents a newer strand of the K-6 curriculum, thus allowing for investigation of what teachers know about this topic now, and perhaps how knowledge increases over time, as better curriculum and professional development become available and teachers gain experience in teaching this domain. We have since added geometry items and expanded our measures upward through middle school content.

Once the domains were specified, we invited a range of experts to write assessment items—mathematics educators, mathematicians, professional developers, project staff, and classroom teachers. We asked for items that posed questions related to the situations that teachers face in their daily work, written in multiple-choice format to facilitate the scoring and scaling of large numbers of teacher responses. We strove to produce items that were ideologically neutral; for example, rejecting any items where a "right" answer might indicate an orientation to "traditional" or "reform" teaching. Finally, we defined mathematical content knowledge for teaching as being composed of two key elements: "common" knowledge of mathematics that any well-educated adult should have *and* mathematical knowledge that is "specialized" to the work of teaching and that only teachers need know. We tried to capture both of these elements in our assessment.

To measure common knowledge of mathematics, we developed questions that, while set in teaching scenarios, still require only the understanding held by most adults. Figure 3 presents one such item:

> Ms. Dominguez was working with a new textbook and she noticed that it gave more attention to the number 0 than her old book. She came across a page that asked students to determine if a few statements about 0 were true or false. Intrigued, she showed them to her sister who is also a teacher, and asked her what she thought.

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Which statement(s) should the sisters select as being true? (Mark YES, NO, or I'M NOT SURE for each item below.)

	Yes	No	I'm not sure
a) 0 is an even number.	1	2	3
b) 0 is not really a number. It is a placeholder in writing big numbers.	1	2	3
c) The number 8 can be written as 008.	1	2	3

Figure 3. Item measuring common content knowledge

To measure the more specialized knowledge of mathematics, we designed items that ask teachers to show or represent numbers or operations using pictures or manipulatives, and to provide explanations for common mathematical rules (e.g., why any number is divisible by 4 if the number formed by the last two digits is divisible by 4).

Figure 4 shows an item that measures specialized content knowledge. In this scenario, respondents evaluate three different approaches to multiplying 35 × 25 and determine whether any of these is a valid general method for multiplication. Any adult should know how to multiply 35 × 25 (see our earlier example), but teachers are often faced with evaluating unconventional student methods that produce correct answers, but whose generalizability or mathematical validity are not immediately clear. For teachers to be effective, they must be able to size up mathematical issues that come up in class—often fluently and with little time.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

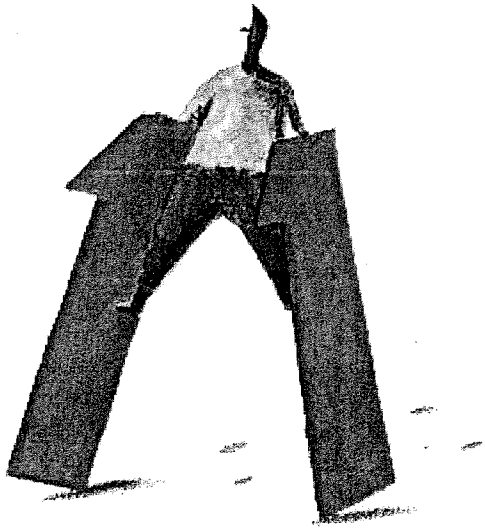
Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+ 700	150
875	875	100
		+ 600
		875

Which of these students is using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 4. Item measuring specialized content knowledge

The claim that we can measure knowledge that is related to high-quality teaching requires solid evidence.



Although students are mentioned in this item, the question does not actually tap respondents' knowledge of students, or of how to teach multiplication to students. Instead, it asks a mathematical question about alternate solution methods, which represents an important skill for effective teaching.

Based on our study of practice as well as the research base on teaching and learning mathematics, analyses of curriculum materials, examples of student work, and personal experience, we have developed over 250 multiple-choice items designed to measure teachers' common and specialized mathematical knowledge for teaching. Many dozens more are under development. Building a good item from start (early idea stage) to finish (reviewed, revised, critiqued, polished, pilot-tested, and analyzed) takes over a year, and is expensive. However we regarded this as an essential investment—a necessary trade-off for the ease, reliability, and economy of a large-scale multiple-choice assessment.

Our aim is to identify the content knowledge needed for effective practice and to build measures of that knowledge that can be used by other researchers. The claim that we can measure knowledge that is related to high-quality teaching requires solid evidence. Most important for our purposes is whether high performance on our items is related to effective instruction. Do teachers' scores on our items predict that they teach with mathematical skill, or that their students learn more, or better?

Is There Knowledge of Mathematics for Teaching? What Do Our Studies Show?

We were fortunate to be involved in a study that would allow us to answer this question. The Study of Instructional Improvement, or SII, is a longitudinal study of schools engaged in comprehensive school reform efforts. As part of that study, we collected student scores on the mathematics portion of the Terra Nova (a reliable and valid standardized test) and calculated a "gain score"—or how many points they gained over the course of a year. We also collected information on these students' family background—in particular their socioeconomic status, or SES—for use in predicting the size of student gain scores. And importantly, we also included many of our survey items—including those in Figures 3 and 4—on the teacher questionnaire. Half of these items measured "common" content knowledge and half measured "specialized" content knowledge. Teachers who participated in the study by answering these questions allowed us to test the relationship between their knowledge for teaching mathematics and the size of their students' gain on the Terra Nova.

The results were clear: In the analysis of 700 first- and third-grade teachers (and almost 3,000 students), we found that teachers' performance on our knowledge for teaching questions—including both common and specialized content knowledge—significantly predicted the size of student gain scores, even though we controlled for things such as student SES, student absence rate, teacher credentials, teacher experience, and average length of mathematics lessons (Hill, Rowan, and Ball, 2005). The students of teachers who answered more items correctly gained more over the course of a year of instruction.

Comparing a teacher who achieved an average score on our measure of teacher knowledge to a teacher who was in the top quartile, the students of the above-average teacher showed gains in their scores that were equivalent to that of an extra two to three weeks of instruction. Moreover, the size of the effect of teachers' mathematical knowledge for teaching was comparable to the size of the effect of socioeconomic status on student gain scores. This was a promising finding because it suggests that improving teachers' knowledge may be one way to stall the widening of the achievement gap as poor children move through school. The research literature on the effect of SES on student achievement indicates that there tends to be a significant achievement gap when students first enter school and that many disadvantaged children fall further and further behind with each year of schooling. Our finding indicates that, while teachers' mathematical knowledge would not by itself overcome the existing achievement gap, it could prevent that gap from growing. Thus, our research suggests that one important contribution we can make toward social justice is to ensure that every student has a teacher who comes to the classroom equipped with the mathematical knowledge needed for teaching.

This result naturally led us to another question: Is teachers' mathematical knowledge distributed evenly across our sample of students and schools, regardless of student race and socioeconomic status? Or, are minority and higher-poverty students taught by teachers with less of this knowledge? Our data show only a very mild relationship between student SES and teacher knowledge, with teachers of higher-poverty students likely to have less mathematical knowledge. The relationship with students' race, however, was stronger. In the third grade, for instance, student minority status and teacher knowledge were negatively correlated, at −.26. That is, higher-knowledge teachers tended to teach non-minority students, leaving minority students with less knowledgeable teachers who are unable to contribute as much to students' knowledge over the course of a year. We find these results shameful. Unfortunately, they are also similar to those found

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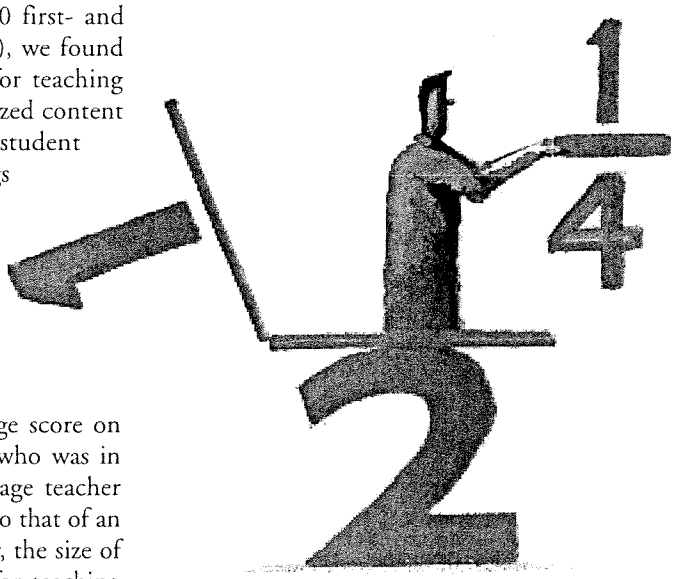

elsewhere with other samples of schools and teachers (Hill and Lubienksi, in press; Loeb and Reininger, 2004). They also suggest that a portion of the achievement gap on the National Assessment of Educational Progress and other standardized assessments might result from teachers with less mathematical knowledge teaching more disadvantaged students. One strategy toward narrowing this gap, then, could be investing in the quality of mathematics content knowledge among teachers working in disadvantaged schools. This suggestion is underscored by the comparable effect sizes of teachers' knowledge and students' socioeconomic status on achievement gains.

Another arena for testing our ideas is in professional development. If there is knowledge of mathematics for teaching, as our studies suggest, then it should be possible for programs to help teachers acquire such knowledge. To probe this, we investigated whether elementary teachers learned mathematical knowledge for teaching in a relatively traditional professional development setting—the summer workshop component of California's K-6 mathematics professional development institutes—and, if so, how much and what those teachers learned (Hill and Ball, 2004). We explored whether our measures of teachers' content knowledge for teaching could be deployed to evaluate a large public program rigorously. We found that teachers did learn content knowledge for teaching mathematics as a result of attending these institutes. We also found that greater performance gains on our measures were related to the length of the institutes and to curricula that focused on proof, analysis, exploration, communication, and representations (Hill and Ball, 2004). In addition to these specific findings, this study set the stage for future analyses of the conditions under which teachers learn mathematical content for teaching most effectively.

One of the most pressing issues currently before us is whether specialized knowledge for teaching mathematics exists independently from common content knowledge—the basic skills that a mathematically literate adult would possess. Analyses of data from large early pilots of our surveys with teachers (Hill and Ball, 2004) suggest that the answer may be yes. Often we found that results for the questions representing "specialized" knowledge of mathematics (e.g., Figure 4) were separable statistically from results on the "common" knowledge items (e.g., Figure 3). In other words, correctly answering the kind of question in Figure 4 seemed to require knowledge over and above that entailed in answering the other kind correctly (e.g., Figure 3). This suggests that there is a place in professional preparation for concentrating on teachers' specialized knowledge. It may even support a claim by the profession to hold a sort of applied mathematical knowledge unique to the work of teaching. If this finding bears out in further research, it strengthens the claim that effective teaching entails a knowledge of mathematics above and beyond what a mathematically literate adult learns in grade school, a liberal arts program, or even a career in another mathematically intensive profession such as accounting or engineering. Professional education of some sort—whether it be pre-service or on the job—would be needed to support this knowledge.

Conclusions

Our work has already yielded tentative answers to some of the questions that drive current debates about education policy and professional practice. Mathematical knowledge for teaching, as we have conceptualized and measured it, does positively predict gains in student achievement (Hill, Rowan, and Ball, 2005). More work remains: Do different kinds of mathematical knowledge for teaching—specialized knowledge or common knowledge, for example, or knowledge of students and content together—contribute more than others to student achievement? The same can be said for building a knowledge base about effective professional development. Historically, most content-focused professional development has been evaluated locally, often with perceptual measures (e.g., do teachers *believe* that they learned mathematics?) rather than true measures of teacher and student learning (see Wilson and Berne, 1999). Developing rigorous measures, and having significant numbers of professional developers use them, will help to build generalizable knowledge about teachers' learning of mathematics. We emphasize that this must be a program of research across a wide sector of the scholarly community; many studies are required in order to make sense of how differences in program content might affect teachers, teaching, and student achievement.

These results represent progress on producing knowledge that is both credible and usable. In the face of this, the negative responses we have received from some other education professionals are noteworthy. Testing teachers, studying teaching or teacher learning, at scale, using standardized student achievement measures—each of these draws sharp criticism from some quarters. Some disdain multiple-choice items, claiming that nothing worth measuring can be measured with such questions. Others argue that teaching, and teacher learning, are such fine-grained complex endeavors that large-scale studies cannot probe or uncover anything worth measuring. Still others claim that we are "deskilling" or "deprofessionalizing" teachers by "testing" them. We argue that these objections run counter to the very core of the critical agenda we face as a professional community.

Until and unless we, as educators, are willing to claim that there is professional knowledge that matters for the quality of instruction and can back that claim with evidence, we will continue to be no more than one voice among many competing to assert what teachers should know and how they might learn that, and why. Our claims to professional knowledge will be no more than the weak claim that we are professionals and deserve authority because we say so, not because we can show that what we know stands apart from what just anyone would know. Isolating aspects of knowing mathematics different from that which anyone who has graduated from sixth grade would know, and demonstrating convincingly that this knowledge matters for students' learning, is to claim *skill* in teaching, not to *deskill* it. Making these arguments, too, is part of the challenge we face as we seek to meet the contemporary challenges to our jurisdictional authority.

Our research group's experience in working from and with problems of professional practice, testing and refining

them with tools that mediate the power of our own convictions and common sense, is one example of the work of trying to build knowledge that is both credible and useful to a range of stakeholders. Many more examples exist and can be developed. Doing so is imperative in the current environment in which demands for education quality are made in a climate of distrust and loss of credibility. Meeting this challenge is a professional responsibility. Doing so successfully is essential to our survival as a profession. □

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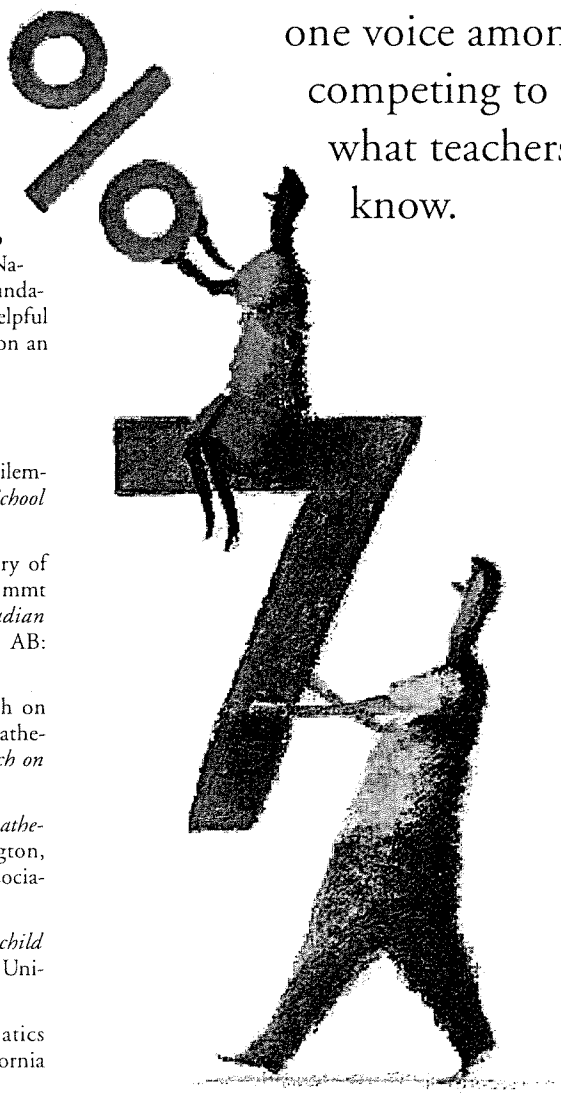

Endnote

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